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by

M. J. P. Cullen

September 1989

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LONDON, METEOROLOGICAL OFFICE.
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03271089 551.509.313 FH2A

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Meteorological Office
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September 1989

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A test of a semi-implicit integration technique for a fully compressible non-hydrostatic model

By M. J. P. CULLEN

Meteorological Office, Bracknell.

SUMMARY

A semi-implicit scheme, in which the time-step is only limited by the advective criterion, is analysed and successfully tested in a fully compressible non-hydrostatic mesoscale model.

1. INTRODUCTION

This note describes an extension of the semi-implicit integration scheme used in the fully compressible non-hydrostatic mesoscale model developed by Tapp and White (1976), henceforth TW. Their scheme treats sound waves implicitly, and the maximum timestep permitted is then given by the inverse of the Brunt-Vaisala frequency. In this note, the scheme is extended to treat gravity waves implicitly, in the way that is now standard for hydrostatic models. This method is analysed by Simmons, Hoskins and Burridge (1978). Once this is done, the time step is only restricted by the advection speed, and could then be further increased by the use of semi-Lagrangian methods.

2. FINITE DIFFERENCE SCHEME

TW, equations (4-6), set out the compressible non-hydrostatic equations using the Exner function π as the pressure variable where $\pi = (P / P_{*})^{\kappa}$, $\kappa = R/c_p$, R is the gas constant, c_p is the specific heat at constant pressure and P_{*} is a constant reference value of the pressure. The variables π and θ are written in terms of deviations π_1 and θ_1 from a steady basic state of the atmosphere $\pi_0(z)$ and $\theta_0(z)$ in hydrostatic balance:

$$\begin{aligned} \pi &= \pi_0(z) + \pi_1(x,y,z,t) \\ \theta &= \theta_0(z) + \theta_1(x,y,z,t) \\ c_p\theta_0\partial\pi_0/\partial z + g &= 0 \end{aligned} \qquad (1)$$

With this transformation, the equations of TW in component form become

$$\begin{aligned} Du/Dt &= fv - c_p(\theta_0 + \theta_1)\partial\pi_1/\partial x + F_x \\ Dv/Dt &= -fu - c_p(\theta_0 + \theta_1)\partial\pi_1/\partial y + F_y \\ Dw/Dt &= g\theta_1/\theta_0 - c_p(\theta_0 + \theta_1)\partial\pi_1/\partial z + F_z \\ D\theta_1/Dt &+ w\partial\theta_0/\partial z = \theta + F_\theta = Q \end{aligned} \qquad (2)$$

$$c_p D\pi_1/Dt = gw/\theta_o - \gamma R(\pi_o + \pi_1)\nabla.\mathbf{v} + \gamma R((\pi_o + \pi_1)/(\theta_o + \theta_1))Q$$

These equations describe the rates of change of the three wind components u, v, w and the perturbation potential temperature and pressure variables θ_1 and π_1.

The basic state (1) is used to extract a linearised version of the sound and gravity wave equations for implicit integration, the remainder of the equations being treated explicitly. Provided that a reasonably representative value of θ_o is chosen, the speed of sound waves is determined to a high degree of accuracy by the temperature distribution of the basic atmosphere itself, since the departures amount to only a few per cent. The choice of a basic atmosphere for semi-implicit integration of the gravity wave terms requires more care, since an inappropriate choice may lead to unconditional computational instability. As shown by Simmons, Hoskins and Burridge (1978), it is necessary to choose a basic atmosphere which is more stable than any likely to be encountered in the real data. An isothermal basic state at 300°K is the standard choice, which we adopt here.

The basic model equations (2) are written in the form:

$$\partial u/\partial t + c_p\theta_o\partial\pi_1/\partial x = -\mathbf{v}.\nabla u + fv - c_p\theta_1\partial\pi_1/\partial x + F_x = X_E$$

$$\partial v/\partial t + c_p\theta_o\partial\pi_1/\partial y = -\mathbf{v}.\nabla v - fu - c_p\theta_1\partial\pi_1/\partial y + F_y = Y_E$$

$$\partial w/\partial t + c_p\theta_o\partial\pi_1/\partial z - g\theta_1/\theta_o = -\mathbf{v}.\nabla w - c_p\theta_1\partial\pi_1/\partial z + F_z = Z_E \qquad (3)$$

$$\partial\theta_1/\partial t + w\partial\theta_o/\partial z = -\mathbf{v}.\nabla\theta_1 + Q = S_E$$

$$\partial\pi_1/\partial t - gw/(c_p\theta_o) + (\gamma-1)\pi_o\nabla.\mathbf{v} = P_E$$

$$P_E = -\mathbf{v}.\nabla\pi_1 - (\gamma-1)\pi_1\nabla.\mathbf{v} + (\gamma-1)Q((\pi_o + \pi_1)/(\theta_o + \theta_1))$$

The implicit terms appear on the left-hand sides and the explicit terms on the right-hand sides of the equations. Note the separation of the pressure gradient, vertical advection of potential temperature, and divergence terms

into those parts that are present in the basic state atmosphere, which are treated implicitly, and those parts that depend on the perturbations θ_1 and π_1 which are treated explicitly. Note also that both the primary components of the hydrostatic balance are treated implicitly. The implicit terms are represented in finite difference form as a weighted average between time levels n+1 and n-1 so that the finite difference approximations in time to equations (6) are

$$
\begin{aligned}
u^{(n+1)} - u^{(n-1)} &= -2\delta t c_p \theta_0 (\alpha \partial \pi_1^{(n+1)}/\partial x + (1-\alpha)\partial \pi_1^{(n-1)}/\partial x) + 2\delta t X'_E \\
v^{(n+1)} - v^{(n-1)} &= -2\delta t c_p \theta_0 (\alpha \partial \pi_1^{(n+1)}/\partial y + (1-\alpha)\partial \pi_1^{(n-1)}/\partial y) + 2\delta t Y'_E \\
w^{(n+1)} - w^{(n-1)} &= -2\delta t c_p \theta_0 (\alpha \partial \pi_1^{(n+1)}/\partial z + (1-\alpha)\partial \pi_1^{(n-1)}/\partial z) + \\
&\quad 2\delta t g/\theta_0 (\alpha \theta_1^{(n+1)} + (1-\alpha)\theta_1^{(n-1)}) + 2\delta t Z'_E \qquad (4) \\
\theta_1^{(n+1)} - \theta_1^{(n-1)} &= -2\delta t \partial \theta_0/\partial z (\alpha w^{(n+1)} + (1-\alpha) w^{(n-1)}) + 2\delta t S'_E \\
\pi_1^{(n+1)} - \pi_1^{(n-1)} &= 2\delta t g/(c_p \theta_0)(\alpha w^{(n+1)} + (1-\alpha) w^{(n-1)}) \\
&\quad -2\delta t (\gamma-1)\pi_0 (\alpha \nabla . \mathbf{v}^{(n+1)} + (1-\alpha)\nabla . \mathbf{v}^{(n-1)}) + 2\delta t P'_E
\end{aligned}
$$

where α is the weighting factor for the forward timestep.

X'_E, Y'_E, Z'_E, S'_E and P'_E contain the explicit terms which are evaluated at time level n apart from the diffusion terms in F_x, F_y, F_z and F_θ which are evaluated at time level n-1 or implicitly for stability.

The manipulation of these equations is simplified by introducing a 2nd order correction (in time) field Π where

$$\Pi^n = 2c_p(\alpha \pi_1^{(n+1)} - \pi_1^{(n)} + (1-\alpha)\pi_1^{(n-1)}) \qquad (5)$$

Reordering (5) to give an equation for $\pi_1^{(n+1)}$ and substituting for $\pi_1^{(n+1)}$ in (4) gives

$$
\begin{aligned}
u^{(n+1)} &= u^{(n-1)} - \delta t \theta_0 \partial \Pi^n/\partial x + 2\delta t (X'_E - c_p \theta_0 \partial \pi_1^{(n)}/\partial x) \\
v^{(n+1)} &= v^{(n-1)} - \delta t \theta_0 \partial \Pi^n/\partial y + 2\delta t (Y'_E - c_p \theta_0 \partial \pi_1^{(n)}/\partial y) \\
w^{(n+1)} &= w^{(n-1)} - \delta t \theta_0 \partial \Pi^n/\partial z + 2\delta t g/\theta_0 (\alpha \theta_1^{(n+1)} + (1-\alpha)\theta_1^{(n-1)}) \\
&\quad + 2\delta t (Z'_E - c_p \theta_0 \partial \pi_1^{(n)}/\partial z) \qquad (6)
\end{aligned}
$$

$$\theta_1^{(n+1)} = \theta_1^{(n-1)} - 2\delta t \partial\theta_o/\partial z(\alpha w^{(n+1)} + (1-\alpha)w^{(n-1)}) + 2\delta t S'_E$$

$$\pi_1^{(n+1)} = (\pi_1^{(n)} - (1-\alpha)\pi_1^{(n-1)})/\alpha + \Pi^n/(2c_p\alpha)$$

where

$$\Pi^n = - 4c_p\alpha\delta t(-g/(c_p\theta_o)(\alpha w^{(n+1)} + (1-\alpha)w^{(n-1)}))$$
$$- 4c_p\alpha\delta t(\gamma-1)\pi_o(\alpha\nabla.\mathbf{v}^{(n+1)} + (1-\alpha)\nabla.\mathbf{v}^{(n-1)}) \qquad (7)$$
$$- 4c_p\alpha\delta t\phi$$

and

$$\phi = -P'_E + (\pi_1^{(n)} - \pi_1^{(n-1)})/(2\alpha\delta t)$$

The next step is to eliminate $\theta^{(n+1)}$ from (6), giving

$$w^{(n+1)}\{1 + 4\alpha^2\delta t^2 g/\theta_o\partial\theta_o/\partial z\} = w^{(n-1)} - \qquad (8)$$
$$\delta t\theta_o\partial\Pi^n/\partial z + 2\delta t g/\theta_o\{\theta_1^{(n-1)} + 2\alpha\delta t S'_E - 2\alpha(1-\alpha)\delta t w^{(n-1)}\partial\theta_o/\partial z\} +$$
$$2\delta t(Z'_E - c_p\theta_o\partial\pi_1^{(n)}/\partial z)$$

Eliminating the unknown fields $u^{(n+1)}, v^{(n+1)}$ and $w^{(n+1)}$ from the equation for Π^n gives the Helmholtz equation

$$\nabla^2_H\Pi^n + (\partial/\partial z(A^{-1}\partial/\partial z) - (g/c_o^2)A^{-1}\partial/\partial z - 1/(4\alpha^2c_o^2\delta t^2))\Pi^n = \Phi \qquad (9)$$

where

$$A^{-1} = (1 + 4\alpha^2\delta t^2 g/\theta_o\partial\theta_o/\partial z)^{-1}.$$

The Helmholtz equation is solved with Neumann boundary conditions, obtained by replacing the normal component of $u^{(n+1)}$ at each boundary by a specified value at time level n.

A practical modification to the scheme is made at this point. For full consistency, the eliminations leading to equation (9) should be made using the spatial finite differences from the model, rather than the analytic derivatives set out here. If this is done, A becomes an $(n-1)\times(n-1)$ matrix, where n is the number of model levels. This matrix contains off-diagonal terms which result in the term $4\alpha^2\delta t^2 g/\theta_o\partial\theta_o/\partial z$ being smoothed in the

vertical. Applying the inverse matrix A^{-1} in (9) thus acts as a vertical anti-diffusion, which can cause problems. In practice, the matrix A^{-1} is diagonalised before being used in the model.

The terms in Φ can all be evaluated using values at time levels n and n-1. Once the Helmholtz equation has been solved the future values of u, v and w are given by equations (6). The future value of θ can be calculated once the future value of w has been calculated.

Thus the complete set of equations can be written as

$$
\begin{aligned}
u^{(n+1)} &= u^{(n-1)} - \delta t \theta_0 \partial \Pi^n / \partial x + X \\
v^{(n+1)} &= v^{(n-1)} - \delta t \theta_0 \partial \Pi^n / \partial y + Y \\
A w^{(n+1)} &= w^{(n-1)} - \delta t \theta_0 \partial \Pi^n / \partial z + Z \\
\theta_1^{(n+1)} &= \theta_1^{(n-1)} - 2\alpha \delta t w(n+1) \partial \theta_0 / \partial z + S \\
\pi_1^{(n+1)} &= (\pi_1^{(n)} - (1-\alpha)\pi_1^{(n-1)})/\alpha + \Pi^n/(2c_p\alpha) \\
\nabla^2_H \Pi^n &+ (\partial/\partial z (A^{-1} \partial/\partial z) - (g/c_0^2) A^{-1} \partial/\partial z - 1/(4\alpha^2 c_0^2 \delta t^2)) \Pi^n = \Phi
\end{aligned}
\qquad (10)
$$

The terms X, Y, Z, S and Φ which appear in equations (10) and ϕ from equation (7) are given in the Appendix.

3. STABILITY ANALYSIS

Consider as in TW a linearised version of equations (2) applied to the study of sound and gravity wave propagation in a vertical (x, z) cross-section of the basic adiabatic atmosphere. The equations are then

$$
\begin{aligned}
\partial u/\partial t + c_p \theta_0 \partial \pi_1 / \partial x &= 0 \\
\partial w/\partial t + c_p \theta_0 \partial \pi_1 / \partial z - g\theta_1/\theta_0 &= 0 \\
\partial \theta_1/\partial t + w \partial \theta_0 / \partial z &= 0 \\
c_p \theta_0 \partial \pi_1 / \partial t - gw + c_0^2 (\partial u/\partial x + \partial w/\partial z) &= 0
\end{aligned}
\qquad (11)
$$

where c_0 is the speed of sound in the unperturbed adiabatic atmosphere.

If we eliminate u, π_1 and θ from (11) we obtain an equation for w of the form

$$\partial^4 w/\partial t^4 + M\partial^2 w/\partial t^2 - c_o^2\partial^2/\partial x^2 N^2 w = 0 \qquad (12)$$

where the operator $M = -c_o^2(\partial^2/\partial z^2+\partial^2/\partial x^2) + N^2$ and $N^2=(g/\theta_o)\partial\theta_o/\partial z$. Let μ_r be the eigenvalues of M. For purely vertical oscillations independent of x, these eigenvalues can be written

$$\mu_r^2 = \sigma_r^2+N^2$$

where σ_r is the frequency of the r'th harmonic of the acoustic oscillations of a column of air of depth h. It can be shown (c.f. Lamb 1932) that the frequencies are given approximately by

$$\sigma_r^2 = (r\pi c_o/h)^2, \quad r=0, 1, 2\cdots.$$

In the general case where there is a horizontal dependence of the form e^{ikx}, the eigenvalues μ satisfy the equation

$$\mu^2 = \mu_r^2+c_o^2k^2.$$

Since the maximum value of k is $\pi/\delta x$, we have

$$\mu_r^2/c_o^2k^2 \geq r^2\delta x^2/h^2.$$

The eigenvalues ν_r of equation (12) are then

$$\pm\sqrt{\{-\tfrac{1}{2}\mu_r^2\pm\sqrt{(\tfrac{1}{4}\mu_r^4-c_o^2k^2N^2)}\}} \qquad (13)$$

which are all pure imaginary values. This is in contrast to the situation in TW where M does not include the N^2 term and ν_r may be complex.

Now approximate equations (11) by the finite difference scheme in time used in section 2:

$$\partial u/\partial t \simeq (u(n+1)-u(n-1))/\delta t$$

$$\partial\pi_1/\partial x \simeq \alpha\partial\pi_1/\partial x(n+1)+(1-\alpha)\partial\pi_1/\partial x(n-1)$$

Write

$$f(\lambda) = (\lambda-1)/(1+(\lambda-1)\alpha)$$

where λ is the amplification factor between time levels (n-1) and (n+1). Assume x variations proportional to e^{ikx}, and leave the vertical variations analytic for the time being. Then (11) becomes

$$f(\lambda)u + c_p\theta_o ik\pi_1(1+(\lambda-1)\alpha) = 0$$

$$f(\lambda)w + c_p\theta_o\partial\pi_1/\partial z - g\theta_1/\theta_o = 0$$

$$f(\lambda)\theta + w\partial\theta_o/\partial z = 0$$

$$c_p\theta_o(\lambda-1)\pi_1 - gw + c_o^2(iku+\partial w/\partial z) = 0$$

Eliminate u, θ and π_1 from this set. An analysis of the vertical finite difference scheme requires this elimination to be performed at finite difference level. This gives

$$f(\lambda)^4 w + (-c_o^2(\partial^2/\partial z^2-k^2)+N^2+g\partial/\partial z)wf^2(\lambda) - N^2w = 0 \qquad (14)$$

Let M now be the finite difference matrix representation of

$$-c_o^2(\partial^2/\partial z^2-k^2)+N^2+g\partial/\partial z.$$

Equation (14) becomes the set of finite difference equations

$$f(\lambda)^4 w + Mf^2(\lambda)w + c_o^2k^2N^2w = 0 \qquad (15)$$

with one equation for each level in the vertical. The vertical grid staggering used ensures that M has strictly positive eigenvalues μ_r. This is true whether or not the diagonalisation of the matrix A^{-1} described in section 2 is employed. Multiply (15) by the transpose of the matrix of eigenvectors of M

$$f(\lambda)^4\omega + \mu_r f^2(\lambda)\omega + c_o^2N^2\omega = 0 \qquad (16)$$

where there is now one equation for each eigenvalue. The eigenvalues ν_r of (16) are all pure imaginary, as with the continuous equation (13). λ then satisfies

$$\lambda = (1+\nu_r-\alpha\nu_r)/(1-\alpha\nu_r).$$

Since ν_r is pure imaginary, $|\lambda|^2$ is given by

$$\{(1+\alpha\nu_r^2-\alpha^2\nu_r^2)^2-\nu_r^2\}/(1-\alpha^2\nu_r^2)^2.$$

If $\alpha=\frac{1}{2}$, this gives $|\lambda|^2=1$. If $\frac{1}{2}\leq\alpha\leq 1$, then $|\lambda|^2\leq 1$ for any ν_r, giving computational stability.

In the scheme described by TW, the aspect ratio of the grid is

significant in determining stability, and the overall stability criterion is $N^2\delta t^2 \leq 1$.

4. RESULTS

The scheme is illustrated by results from the quasi-operational version of the TW model described by Golding (1987). The dynamical representation in this model is more complicated than in TW to allow for orography, this extension is described by Carpenter (1979). The new integration scheme carries over directly to this representation. The vertical velocity w is replaced by a vertical velocity $\dot{\eta}$ in a transformed coordinate. It is found essential that the upper boundary condition for the model specifies $\dot{\eta}$. This configuration of the model has a 15km grid in the horizontal and 16 vertical levels with the highest resolution near the ground. The timestep restriction using the scheme given by TW is about 45 seconds for real data and the vertical resolution used. The most critical areas are the upper troposphere and the surface layer where high static stabilities can occur. The model is normally run with a 60 second timestep by using strong time filtering and a backward weighting of the implicit time integration used for the sound waves.

The scheme described in this paper is implemented with the same model configuration and a timestep increased to 120 seconds. The performance is illustrated by two cases. In the first, for 15 February 1989, (Fig. 1), high static stability in the surface layer led to instability of the TW integration scheme with a 60 second timestep. The effect is clearly seen in the screen level temperature field illustrated. This is removed using the new scheme. In the second case, for 16 October 1987, (Fig. 2), the use of the extended timestep is shown to have little effect on the quality of a forecast of extreme synoptic development with exceptional winds. This

illustrates that the possibly increased time truncation errors in handling gravity waves in the new scheme, and the use of a specified $\dot{\eta}$ at the upper boundary does not have any significant impact on the results.

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APPENDIX

The terms X, Y, Z, S, ϕ and Φ which appear in the finite difference scheme given in section 2 are

$$X = 2\delta t\,(fv^{\langle n\rangle} - \mathbf{v}^{\langle n\rangle}.\nabla u^{\langle n\rangle} - c_p(\theta_0+\theta_1^{\langle n\rangle})\partial\pi_1^{\langle n\rangle}/\partial x + F_x)$$

$$Y = 2\delta t\,(-fu^{\langle n\rangle} - \mathbf{v}^{\langle n\rangle}.\nabla v^{\langle n\rangle} - c_p(\theta_0+\theta_1^{\langle n\rangle})\partial\pi_1^{\langle n\rangle}/\partial y + F_y)$$

$$Z = 2\delta t g/\theta_0(\theta_1^{\langle n-1\rangle} + \alpha S - 2\alpha(1-\alpha)\delta t w^{\langle n-1\rangle}\partial\theta_0/\partial z) +$$
$$2\delta t\,(- \mathbf{v}^{\langle n\rangle}.\nabla w^{\langle n\rangle} - c_p(\theta_0+\theta_1^{\langle n\rangle})\partial\pi_1^{\langle n\rangle}/\partial z + F_z)$$

$$S = 2\delta t\,(-(1-\alpha)w^{\langle n-1\rangle}\partial\theta_0/\partial z - \mathbf{v}^{\langle n\rangle}.\nabla\theta_1^{\langle n\rangle} + Q)$$

$$\phi = \mathbf{v}^{\langle n\rangle}.\nabla\pi_1^{\langle n\rangle} + (\gamma-1)\pi_1^{\langle n\rangle}\nabla.\mathbf{v}^{\langle n\rangle} - (\gamma-1)Q((\pi_0 + \pi_1^{\langle n\rangle})/(\theta_0 + \theta_1^{\langle n\rangle}))$$
$$+ (\pi_1^{\langle n\rangle} - \pi_1^{\langle n-1\rangle})/(2\alpha\delta t)$$

$$\Phi = (1/(\alpha\theta_o\delta t))(\nabla.\mathbf{v}^{(n-1)} + (A^{-1}-I)\partial w^{(n-1)}/\partial z +$$

$$\alpha(\partial X/\partial x+\partial Y/\partial y+A^{-1}\partial Z/\partial z))$$

$$- (g/c_o^2)((1-\alpha)w^{(n-1)}+\alpha A^{-1}(w^{(n-1)}+Z) + (c_p\theta_o/c_o^2)\phi)$$

FIGURE CAPTIONS

Fig. 1. Screen level temperature (°C) and 10m wind (kt) forecast for 06 hr on 16 February 1989.

a) TW scheme with 60 s timestep.

b) Present scheme with 120 s timestep.

Fig. 2. Surface pressure (hpa), 10m wind (kt) and precipitation forecast for 06 hr on 16 October 1987. Open and filled circles represent light and heavy rain, open and filled triangles represent light and heavy showers.

a) TW scheme with 60 s timestep.

b) Present scheme with 120s timestep.

FIG. (1A)

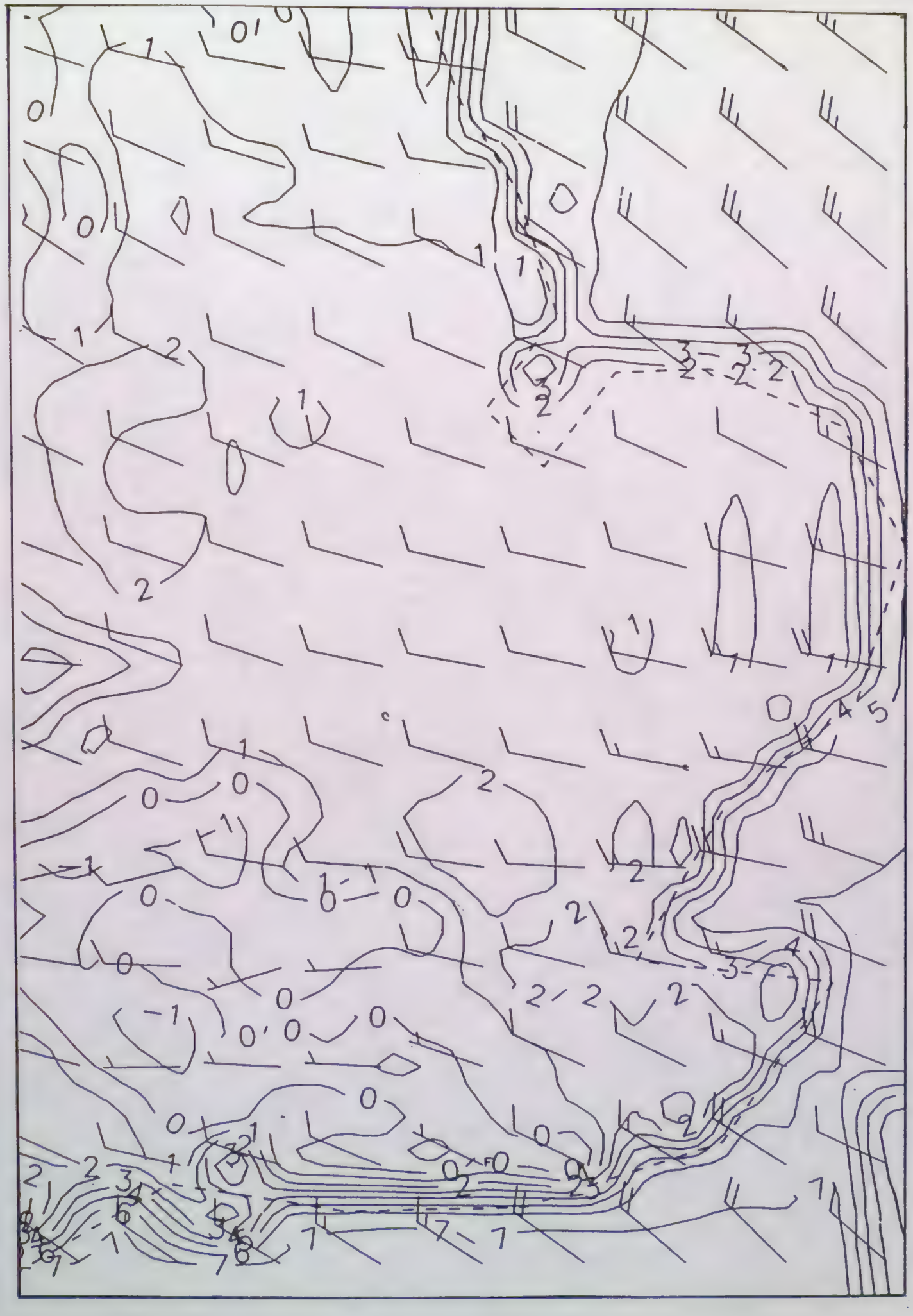

FIG. (1B)

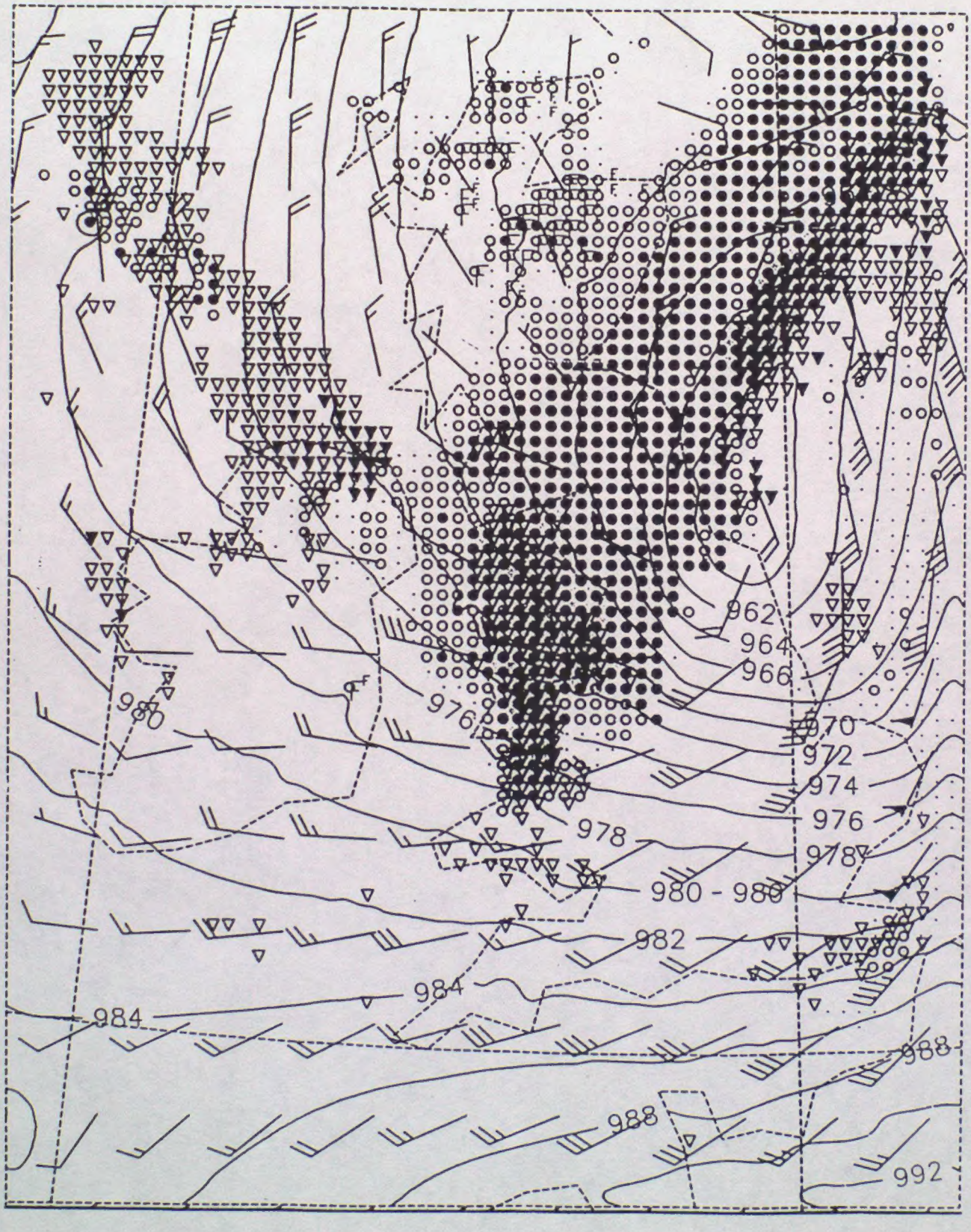

FIG. (2A)

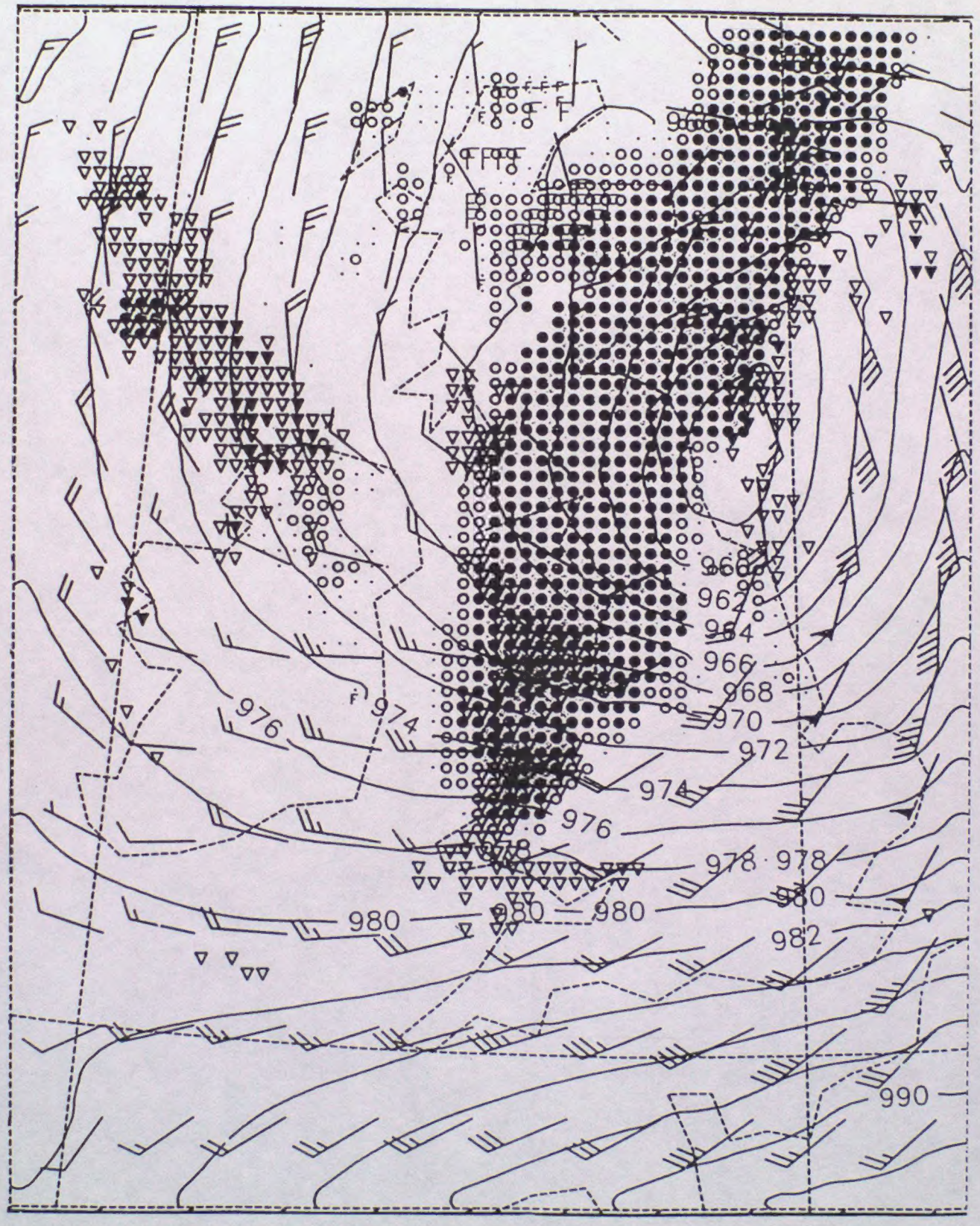
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FIG. (2B)

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